The Laplace Transform Approach to Linear Transmission Line Analysis: Wave Variables

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1. Introduction

In an earlier report[1] the Laplace domain form of the chain matrix description for a *lossy* discretized transmission line was presented. It was noted that this is not the only means to describe the behavior of a transmission line. However, we shall see here that the chain matrix description can be used to derive other models for the behavior of discretized (sectionalized) transmission lines. In particular, this report presents the *wave variable* description. In NDT18-09-2007 it was noted that Burkhart and Wilcox [1] used the lossless version of the theory as a basis for transmission line synthesis to realize pulse shapes. The work in [1] was directly founded upon the inverse scattering theory of Bruckstein and Kailath [2]. However, in this report we initially follow the more general approach in Frolik and Yagle [3] which accounts for the case of lossy and dispersive lines[2]. The present report and NDT18-09-2007 pull together these varied accounts of transmission line modeling based around transform methods and line discretization.

Therefore, in Section 2 we present the detailed derivation of the wave variable description for a single uniform and lossy line section. In Section 3 the result of Section 2 is specialized to the lossless line case for ease of exposition. Section 3 completes the basic derivation of the discretized transmission line wave variable descriptions and so concludes this report. The application of the models of this report to solving transmission line inverse scattering problems or transmission line synthesis problems is considered elsewhere.

2. The Wave Variable Description for the Discretized Lossy Transmission Line

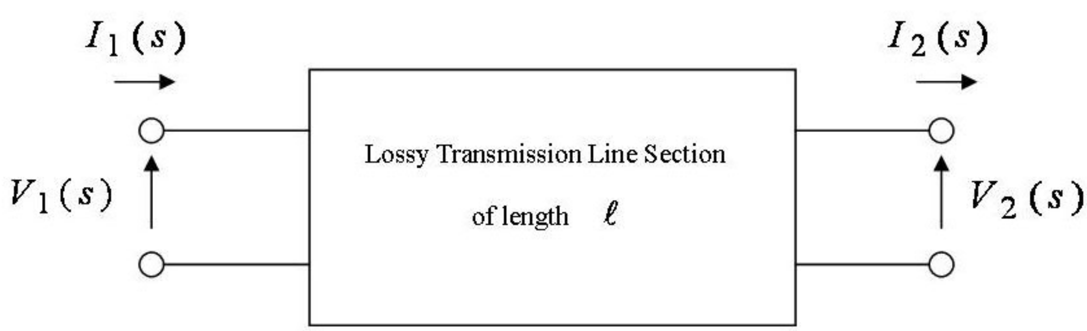

Figure 1: Block diagram representation of a uniform section of lossy transmission line of length ℓ with the Laplace domain port currents and voltages as indicated.

[1] "The Laplace Transform Approach to Linear Transmission Line Analysis," Technical Report NDT18-09-2007, 11 September 2007, NanoDotTek (15 pages).

[2] It is worth noting that the line model in [3] is not completely general since a smoothly nonuniform line is modeled as the cascade interconnection of uniform sections, which is discretization. Moreover, not all physical effects are modeled. For example, the skin effect is not considered in [3] or in this report either. Radiative losses at open line terminations are also not modeled.

We begin with a quick review of the main results from NDT18-09-2007. Figure 1 (above) depicts the block diagram representation of a uniform section of a lossy transmission line of length ℓ. It was shown that the port voltages and currents (Laplace domain) are related according to

$$\begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma(s)\ell) & -Z(s)\sinh(\gamma(s)\ell) \\ -Y(s)\sinh(\gamma(s)\ell) & \cosh(\gamma(s)\ell) \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} \tag{2.1}$$

(outputs in terms of inputs) or else by

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma(s)\ell) & Z(s)\sinh(\gamma(s)\ell) \\ Y(s)\sinh(\gamma(s)\ell) & \cosh(\gamma(s)\ell) \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} \tag{2.2}$$

(inputs in terms of outputs). As well, the *propagation constant* and *characteristic impedance* of the line section are, respectively,

$$\gamma(s) = \sqrt{(Ls+R)(Cs+G)} \text{ , and} \tag{2.3}$$

$$Z(s) = \frac{V_R(s)}{I_R(s)} = \frac{V_L(s)}{I_L(s)} = \sqrt{\frac{Ls+R}{Cs+G}} = \frac{1}{Y(s)} \tag{2.4}$$

It is also important to recall the structure of the solution to the wave equation for the transmission line:

$$\begin{aligned} V(x,s) &= V_R(s)e^{-\gamma(s)x} + V_L(s)e^{\gamma(s)x} \\ I(x,s) &= I_R(s)e^{-\gamma(s)x} - I_L(s)e^{\gamma(s)x} \end{aligned} \tag{2.5}$$

Terms involving functions with subscript "R" correspond to right-propagating waves, while terms involving functions with subscript "L" correspond to left-propagating waves according to the convention described in NDT18-09-2007. This is something we expand upon now.

We now define the *right-propagating*, and *left-propagating wave variables* at location x on the transmission line as

$$\begin{aligned} W_R(x,s) &= \frac{1}{2}[V(x,s) + Z(s)I(x,s)] \text{, } \mathit{and} \\ W_L(x,s) &= \frac{1}{2}[V(x,s) - Z(s)I(x,s)] \end{aligned} \tag{2.6}$$

respectively. These definitions are not quite the same as those which appear in either [2], or [3]. However, we will relate the present definitions to those of [2], and [3] later on.

We begin with the definitions of (2.6) as they are somewhat easier to understand at the outset. However, in a certain mathematical sense they are not quite as convenient as those in [2] and [3]. Again, this will be explained below.

From (2.5) and (2.4) substituted into (2.6) we easily arrive at

$$\begin{aligned} W_R(x,s) &= V_R(s)e^{-\gamma(s)x} \\ W_L(x,s) &= V_L(s)e^{\gamma(s)x} \end{aligned} \tag{2.7}$$

If for the moment we consider the special instance of the lossless line and let $s = j\omega$ then we see that $W_R(x,j\omega) = V_R(j\omega)e^{-j\sqrt{LC}\,\omega x}$ (here $\gamma(s) = \sqrt{LC}s$). If we assume monochromatic wave propagation on the line then we may assume that $V_R(j\omega)$ corresponds to (via the inverse Fourier transform) $V_R e^{j\omega t}$ in the time-domain. Thus,

$$w_R(x,t) = V_R \exp[j\omega(t - x/c)] \tag{2.8}$$

($c = 1/\sqrt{LC}$ is the speed of propagation of a wave on the line section)[3]. Equation (2.8) is certainly that of a right-propagating *voltage wave* on the line section. Similar reasoning establishes that $W_L(x,s)$ is the Laplace domain representation for a left-propagating voltage wave on the line section.

Now, in matrix-vector form (2.6) becomes

$$\begin{bmatrix} W_R(x,s) \\ W_L(x,s) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & Z(s) \\ 1 & -Z(s) \end{bmatrix}\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} \tag{2.9}$$

Alternatively

$$\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Z^{-1}(s) & -Z^{-1}(s) \end{bmatrix}\begin{bmatrix} W_R(x,s) \\ W_L(x,s) \end{bmatrix} \tag{2.10}$$

We may generalize the matrix-vector equation in (2.1) to a line possessing *M* sections. Thus, the matrix-vector equation relating the Laplace domain input and output port voltages and currents for the *kth* line section is

$$\begin{bmatrix} V_{k+1}(s) \\ I_{k+1}(s) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma_k(s)\ell_k) & -Z_k(s)\sinh(\gamma_k(s)\ell_k) \\ -Y_k(s)\sinh(\gamma_k(s)\ell_k) & \cosh(\gamma_k(s)\ell_k) \end{bmatrix}\begin{bmatrix} V_k(s) \\ I_k(s) \end{bmatrix} \tag{2.11}$$

[3] We may also recall that frequency (ω) and wavelength (λ) are related according to $\omega = \beta c = \frac{2\pi}{\lambda}c$.

This is for $k = 1,2,\dots,M-1,M$. The *kth* section of line has its input at some distance Δ from the input end of the line at $x = 0$. Similarly, the output end of this section is at a distance $\Delta + \ell_k$ from the input. Define $x^+ = x + \varepsilon$ ($\varepsilon > 0$ and is small). Now define

$$\begin{aligned} W_{R,k}(s) &= W_R(\Delta^+, s), W_{L,k}(s) = W_L(\Delta^+, s) \\ W_{R,k+1}(s) &= W_R((\Delta + \ell_k)^+, s), W_{L,k+1}(s) = W_L((\Delta + \ell_k)^+, s) \end{aligned} \tag{2.12}$$

Because each line section is generally different from the others there will often be discontinuities in the line parameters at the places where the sections are joined together. However, the port voltages and currents must be *continuous* across all such discontinuities. The wave variables of (2.12) are those at infinitesimal distances to the right of these line discontinuities. From (2.9) and (2.10) we therefore find that

$$\begin{bmatrix} V_k(s) \\ I_k(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Z_k^{-1}(s) & -Z_k^{-1}(s) \end{bmatrix} \begin{bmatrix} W_{R,k}(s) \\ W_{L,k}(s) \end{bmatrix} \tag{2.13}$$

and that

$$\begin{bmatrix} W_{R,k+1}(s) \\ W_{L,k+1}(s) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & Z_{k+1}(s) \\ 1 & -Z_{k+1}(s) \end{bmatrix} \begin{bmatrix} V_{k+1}(s) \\ I_{k+1}(s) \end{bmatrix} \tag{2.14}$$

If we now substitute (2.13) and (2.14) into (2.11) we initially discover that

$$\begin{aligned} \begin{bmatrix} W_{R,k+1}(s) \\ W_{L,k+1}(s) \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & Z_{k+1}(s) \\ 1 & -Z_{k+1}(s) \end{bmatrix} \\ &\times \begin{bmatrix} \cosh(\gamma_k(s)\ell_k) & -Z_k(s)\sinh(\gamma_k(s)\ell_k) \\ -Y_k(s)\sinh(\gamma_k(s)\ell_k) & \cosh(\gamma_k(s)\ell_k) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ Z_k^{-1}(s) & -Z_k^{-1}(s) \end{bmatrix} \begin{bmatrix} W_{R,k}(s) \\ W_{L,k}(s) \end{bmatrix} \end{aligned} \tag{2.15}$$

Fortunately, this cumbersome expression reduces (after some algebraic manipulation) to

$$\begin{bmatrix} W_{R,k+1}(s) \\ W_{L,k+1}(s) \end{bmatrix} = \frac{Z_{k+1}(s) + Z_k(s)}{2Z_k(s)} \begin{bmatrix} 1 & -\dfrac{Z_{k+1}(s) - Z_k(s)}{Z_{k+1}(s) + Z_k(s)} \\ -\dfrac{Z_{k+1}(s) - Z_k(s)}{Z_{k+1}(s) + Z_k(s)} & 1 \end{bmatrix} \begin{bmatrix} e^{-\gamma_k(s)\ell_k} & 0 \\ 0 & e^{\gamma_k(s)\ell_k} \end{bmatrix} \begin{bmatrix} W_{R,k}(s) \\ W_{L,k}(s) \end{bmatrix} \tag{2.16}$$

We may now define the *kth reflection coefficient* to be

$$K_k(s) = \frac{Z_{k+1}(s) - Z_k(s)}{Z_{k+1}(s) + Z_k(s)} \tag{2.17}$$

The goal here is to replace (2.16) with an even more compact expression in terms of the reflection coefficient. This is easy to do for the first matrix on the right-hand side of (2.16), but the coefficient $(Z_{k+1}(s) + Z_k(s))/(2Z_k(s))$ cannot be expressed solely in terms of $K_k(s)$. This difficulty has arisen because of the definitions in (2.6). And it is this problem which caused the authors [2,3] to *instead* define the wave variables to be

$$\begin{bmatrix} \hat{W}_{R,k}(s) \\ \hat{W}_{L,k}(s) \end{bmatrix} = Z_k^{-1/2}(s) \begin{bmatrix} W_{R,k}(s) \\ W_{L,k}(s) \end{bmatrix} \tag{2.18}$$

(We employ a hat "^" to distinguish the wave variables in [2] or [3] from the wave variables of (2.6).) These are sometimes called *energy-normalized wave variables.* If we now rewrite (2.16) in terms of the energy normalized wave variables we see that

$$\begin{bmatrix} \hat{W}_{R,k+1}(s) \\ \hat{W}_{L,k+1}(s) \end{bmatrix} = \frac{Z_{k+1}(s)+Z_k(s)}{2\sqrt{Z_k(s)Z_{k+1}(s)}} \begin{bmatrix} 1 & -\frac{Z_{k+1}(s)-Z_k(s)}{Z_{k+1}(s)+Z_k(s)} \\ -\frac{Z_{k+1}(s)-Z_k(s)}{Z_{k+1}(s)+Z_k(s)} & 1 \end{bmatrix} \begin{bmatrix} e^{-\gamma_k(s)\ell_k} & 0 \\ 0 & e^{\gamma_k(s)\ell_k} \end{bmatrix} \begin{bmatrix} \hat{W}_{R,k}(s) \\ \hat{W}_{L,k}(s) \end{bmatrix}$$

(2.19)

It is not hard to verify that

$$\frac{1}{\sqrt{1-K_k^2(s)}} = \frac{Z_{k+1}(s) + Z_k(s)}{2\sqrt{Z_k(s)Z_{k+1}(s)}} \tag{2.20}$$

Immediately, (2.19) becomes

$$\begin{bmatrix} \hat{W}_{R,k+1}(s) \\ \hat{W}_{L,k+1}(s) \end{bmatrix} = \frac{1}{\sqrt{1-K_k^2(s)}} \begin{bmatrix} 1 & -K_k(s) \\ -K_k(s) & 1 \end{bmatrix} \begin{bmatrix} e^{-\gamma_k(s)\ell_k} & 0 \\ 0 & e^{\gamma_k(s)\ell_k} \end{bmatrix} \begin{bmatrix} \hat{W}_{R,k}(s) \\ \hat{W}_{L,k}(s) \end{bmatrix} \tag{2.21}$$

This is the *transmission matrix* description of the relationship between the input and output (energy normalized) wave variables. Equation (2.21) is essentially a generalization of Equation (2.14) in Bruckstein and Kailath [2]. It is a version of Equation (5.4) in Frolik and Yagle [3].

3. The Special Case of the Discretized Lossless Transmission Line

Equation (2.21) is fairly general and correspondingly difficult to work with. Therefore, as this report is intended to be more "introductory" we will simplify matters by assuming that our discretized transmission line is now lossless.

Since the transmission line is now lossless we have $R_k = G_k = 0$ for all of the line sections. Therefore, the section k propagation constant is $\gamma_k(s) = \sqrt{L_k C_k}\, s$ and the characteristic impedance is $Z_k(s) = Z_k = \sqrt{\dfrac{L_k}{C_k}}$ (frequency-independent). The one-way propagation delay for the *kth* line section is therefore, $\tau_k = \sqrt{L_k C_k}\,\ell_k$. It is apparent from (2.17) that the reflection coefficient is frequency-independent and so we may drop the explicit dependence upon the complex variable *s* and write K_k. Now, because of the form of the propagation constant we observe that the complex exponentials in (2.21) become $\exp(\pm\tau_k s)$. At this point a further simplification in the transmission line model is possible. We may assume that the lossless but nonuniform line is modeled as the cascade interconnection of uniform and lossless sections each possessing the same one-way propagation delay $\tau_k = \tau$ for all $k = 1, 2, \dots, M-1, M$. In view of all of these simplifications (2.21) is now reduced to

$$\begin{bmatrix} \hat{W}_{R,k+1}(s) \\ \hat{W}_{L,k+1}(s) \end{bmatrix} = \frac{1}{\sqrt{1-K_k^2}} \begin{bmatrix} 1 & -K_k \\ -K_k & 1 \end{bmatrix} \begin{bmatrix} e^{-\tau s} & 0 \\ 0 & e^{+\tau s} \end{bmatrix} \begin{bmatrix} \hat{W}_{R,k}(s) \\ \hat{W}_{L,k}(s) \end{bmatrix} \tag{3.1}$$

The complex exponentials $e^{-\tau s}$ and $e^{\tau s}$ correspond to analog τ second time-delay and time-advance Laplace domain operators, respectively. The presence of the Laplace domain time-advance operator in (3.1) implies that the model (3.1) is *non-causal.* Of course, this non-causality holds for the more general (2.21) as well.

It is possible to rewrite (3.1) in *scattering matrix* form. This representation is also a *causal* representation. By algebraically manipulating the equations in (3.1) we obtain

$$\begin{bmatrix} \hat{W}_{R,k+1}(s) \\ \hat{W}_{L,k}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\tau s} \end{bmatrix} \begin{bmatrix} \sqrt{1-K_k^2} & -K_k \\ K_k & \sqrt{1-K_k^2} \end{bmatrix} \begin{bmatrix} e^{-\tau s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{W}_{R,k}(s) \\ \hat{W}_{L,k+1}(s) \end{bmatrix} \tag{3.2}$$

We see that there are no time-advance operators present in (3.2) and so the model is now in a causal form. Figure 2 (below) illustrates that waves incident upon the transmission line section are regarded as inputs (lines having arrows pointing into the box), while those waves reflected from the transmission line section are regarded as outputs (lines having arrows pointing away from the box). Equation (3.2) is really Equation (2.22) in [2].

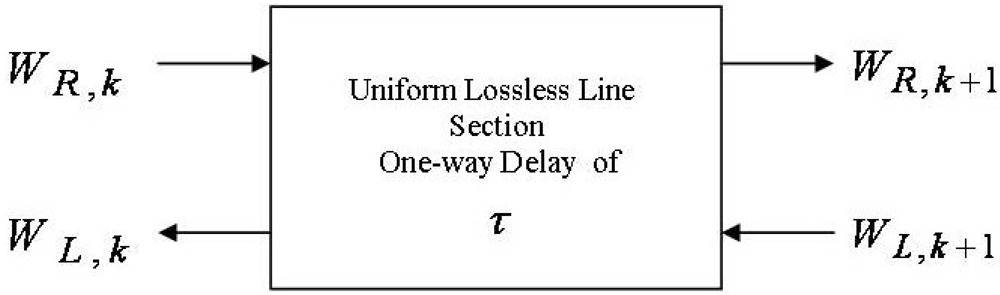

Figure 2: An illustration of the wave variables incident upon and reflected from the lossless line section.

Acknowledgment

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References

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