**Numerical Inversion of the Laplace Transform:
The Methods of Lee-Weeks and of Dubner-Abate**

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1. Introduction

This report revisits certain "old" methods of numerically inverting the Laplace transform. Specifically, the algorithms of Lee-Weeks [1,2], and of Dubner-Abate [3] are considered. The consideration of these algorithms here is motivated by the problems of electrical transmission line transient and steady-state response simulation. Algorithms for the numerical simulation of transmission line responses may be employed in the simulation of reflectometry measurements for the purpose of non-destructively detecting and locating faults on a line. This is to facilitate the development of good signal processing algorithms for fault detection and location. For example, Chang [4] advocates the use of the Lee-Weeks algorithm in the numerical computation of transmission line responses in a very general situation, which is that of nonuniform, coupled, lossy transmission lines with frequency-dependent parameters. In transmission line response simulation the Laplace transforms are irrational functions of the complex variable s, and so a general-purpose numerical method is essential to successfully solve the problem. Of course, many other applications may give rise to the need to invert Laplace transforms. For instance, Springer [5] suggests using the Dubner-Abate algorithm for applications involving the determination of probability density functions in the algebra of random variables.

Many readers may find it difficult to obtain copies of [1],[2] or [3], and so this report contains a fairly detailed summary of the algorithm derivations. This also includes the correction of certain errors which will be pointed out in the appropriate places. The methods considered here date back to the 1960s and earlier. It is quite possible that better approaches exist [1], but the old methods have formed the foundation for more modern approaches. And the old methods at least have the virtue of being easy to understand, are easy to program, and yet are still capable of giving good results depending on the application, and provided that they are used with some discretion.

2. The Algorithm of Dubner-Abate

It seems useful to begin with a review of some basic facts about the Laplace transform as these will be important in subsequent developments.

We will assume that $f(t) \in \mathbf{R}$ (i.e., $f(t)$ is real-valued), is defined for all "time" $t \in [0, \infty) \subset \mathbf{R}$, and that $f(t)$ has growth that is adequately limited to permit the following improper integral to exist

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt \tag{2.1}$$

[1] Indeed, the ACM Transactions on Mathematical Software (TOMS) in the 1990s has various articles about newer methods. For example, there is D. G. Duffy, "On the Numerical Inversion of Laplace Transforms: Comparison of Three New Methods on Characteristic Problems from Applications," ACM Trans. on Math. Software, vol. 19, no. 3, Sept. 1993, pp. 333-359. Also see L. D'Amore, G. Laccetti, A. Murli, "An Implementation of a Fourier Series Method for the Numerical Inversion of the Laplace Transform," ACM Trans. on Math. Software, vol. 25, no. 3, Sept. 1999, pp. 279-305.

where $s = a + i\omega$ ($a, \omega \in \mathbf{R}$, and $i = \sqrt{-1}$). Of course, Equation (2.1) defines the *Laplace transform of* $f(t)$, and $s \in \mathbf{C}$ (complex numbers). The Laplace transform maps a function defined on the non-negative half of the real number line to a complex-valued function defined upon the complex plane. It is not unusual to modify (2.1) by integrating from $-\infty$ to $+\infty$ instead allowing the definition to encompass functions defined upon the entire real number line. The result is often called a *bilateral* transform, whereas (2.1) is *unilateral.* However, the computational problems that motivate the present report prefer to use the definition in (2.1).

From Euler's identity [2]

$$e^{-st} = e^{-at}e^{-i\omega t} = e^{-at}\left[\cos(\omega t) - i\sin(\omega t)\right] \tag{2.2}$$

From (2.2) substituted into (2.1) we observe that

$$F(s) = \underbrace{\int_0^\infty e^{-at}\cos(\omega t) f(t)\, dt}_{= \mathrm{Re}[F(s)]} - i \underbrace{\int_0^\infty e^{-at}\sin(\omega t) f(t)\, dt}_{= -\mathrm{Im}[F(s)]} \tag{2.3}$$

where Re[$\cdot$] denotes "real part of $\cdot$," and Im[$\cdot$] denotes "imaginary part of $\cdot$." From (2.3) it is now easily seen that

$$\mathrm{Re}[F(a + i\omega)] = \mathrm{Re}[F(a - i\omega)]\,, \quad \mathrm{Im}[F(a + i\omega)] = -\mathrm{Im}[F(a - i\omega)] \tag{2.4}$$

The *inverse Laplace transform (ILT)* is

$$f(t) = \frac{1}{2\pi i}\int_{a - i\infty}^{a + i\infty} e^{st}\, F(s)\, ds \tag{2.5}$$

The integration limits imply that we integrate along a vertical line [3] whose x-coordinate (real axis coordinate) is a (with $a > 0$), and that we do so from $-\infty$ to $+\infty$. Thus, a is a constant, and so $ds = i\, d\omega$ in (2.5). It is assumed that $F(s)$ does not possess singularities on the line of integration, or any such to the right of the line. Decomposing $F(s)$ into its real and imaginary parts, and making use of Euler's identity again it is apparent that

$$f(t) = \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}\left[\cos(\omega t)\mathrm{Re}[F(a + i\omega)] - \sin(\omega t)\mathrm{Im}[F(a + i\omega)]\right] d\omega \tag{2.6}$$

[2]Basic material about Euler's identity may be found in many sources such as Zarowski [6]. Some fundamental numerical issues involved in computing exponential functions are considered in [6].

[3]This line is really part of a closed contour on the complex s-plane. The reader ought to consult Springer [5] for more details about this, or else see other sources that discuss the inversion of complex transforms.

There can be no imaginary part in (2.6) since we have assumed that $f(t)$ is real-valued for all time. If we now make use of the conditions in (2.4) we see that (2.6) must reduce to the improper integral

$$f(t) \;=\; \frac{e^{at}}{\pi}\int_0^\infty \cos\,(\omega t)\;\mathrm{Re}[F(a\;+\;i\omega)]\;d\omega \tag{2.7}$$

Following [3] consider a function $h(t)$ assumed to be zero-valued for $t\;<\;0$. This function is regarded as being divided into sections of length T that do not overlap. That is, we consider $h(t)$ on intervals $[nT,(n+1)T)$ for $n\;=\;0,1,2,\;\ldots$. Each such section is reflected through its boundaries to construct an infinite sequence of even and periodic functions denoted by $h_n(t)$ each possessing period $2T$. That is, define

$$h_n(t) \;=\; \begin{cases} h(t) & ,\;\; nT\;\le\;t\;<\;(n+1)T \\ h(2nT-t) & ,\;\; (n-1)T\;\le\;t\;<\;nT \end{cases} \tag{2.8}$$

A Fourier series representation is desired for each of these. Thus, these are rewritten to define them on the interval $(-T,T)$ according to

$$h_n(t) \;=\; \begin{cases} h(nT\;+\;t) & ,\;\; 0\;\le\;t\;<\;T \\ h(nT\;-\;t) & ,\;\; -T\;<\;t\;\le\;0 \end{cases} \tag{2.9a}$$

for $n\;=\;0,2,4,\;\ldots$, and

$$h_n(t) \;=\; \begin{cases} h((n+1)T\;-\;t) & ,\;\; 0\;\le\;t\;<\;T \\ h((n+1)T\;+\;t) & ,\;\; -T\;<\;t\;\le\;0 \end{cases} \tag{2.9b}$$

for $n\;=\;1,3,5,\;\ldots$. Observe that $h_n(t)\;=\;h_n(-t)$ (evenness).

Therefore, a Fourier cosine series expansion is possible according to

$$h_n(t) \;=\; \frac{1}{2}a_{n,0}\;+\;\sum_{k\,=\,1}^{\infty} a_{n,k}\cos\left(\frac{k\pi}{T}t\right) \tag{2.10}$$

where the Fourier series coefficients are given by

$$a_{n,k} \;=\; \frac{2}{T}\int_{nT}^{(n+1)T} h(t)\cos\left(\frac{k\pi}{T}t\right)\;dt \tag{2.11}$$

Substituting (2.11) into (2.10) yields

$$h_n(t) \;=\; \frac{2}{T}\left[\frac{1}{2}\int_{nT}^{(n+1)T} h(x)\;dx\;+\;\sum_{k\,=\,1}^{\infty}\left\{\int_{nT}^{(n+1)T} h(x)\cos\left(\frac{k\pi}{T}x\right)\;dx\right\}\cos\left(\frac{k\pi}{T}t\right)\right] \tag{2.12}$$

Therefore

$$\sum_{n=0}^{\infty} h_n(t) = \frac{2}{T}\left[\frac{1}{2}\int_0^{\infty} h(x)\,dx + \sum_{k=1}^{\infty}\left\{\int_0^{\infty} h(x)\cos\left(\frac{k\pi}{T}x\right)dx\right\}\cos\left(\frac{k\pi}{T}t\right)\right] \tag{2.13}$$

If for $\omega_k = k\pi/T$ we define

$$a(\omega_k) = \int_0^{\infty} h(t)\cos(\omega_k t)\,dt \tag{2.14}$$

then (2.13) becomes

$$\sum_{n=0}^{\infty} h_n(t) = \frac{2}{T}\left[\frac{1}{2}a(\omega_0) + \sum_{k=1}^{\infty} a(\omega_k)\cos(\omega_k t)\right] \tag{2.15}$$

Suppose that

$$h(t) = e^{-at}f(t) \tag{2.16}$$

From (2.14) and recalling the expression for $\mathrm{Re}[F(s)]$ in (2.3) we see that

$$\mathrm{Re}[F(a + i\omega_k)] = \int_0^{\infty} e^{-at}\cos\left(\frac{k\pi}{T}t\right)f(t)\,dt = a(\omega_k) \tag{2.17}$$

Now (2.15) becomes

$$\sum_{n=0}^{\infty} e^{at}\,h_n(t) = \frac{2}{T}e^{at}\left[\frac{1}{2}\mathrm{Re}[F(a)] + \sum_{k=1}^{\infty}\mathrm{Re}[F(a + i\omega_k)]\cos(\omega_k t)\right] \tag{2.18}$$

This arises simply from multiplying (2.15) by $\exp(at)$, and using the association in (2.17). Equation (2.18) is approximately the inverse Laplace transform of $F(s)$ for $t \in [0,T)$.

From (2.9a,b) for $t \in [0,T)$

$$h_n(t) = \begin{cases} h(nT + t) & , \ n = 0,2,4,\ \ldots \\ h((n+1)T - t) & , \ n = 1,3,5,\ \ldots \end{cases} \tag{2.19}$$

Therefore, the left-hand side of (2.18) may be expressed as

$$\sum_{n=0}^{\infty} e^{at}h_n(t) = \sum_{n=0}^{\infty} e^{at}[h_{2n}(t) + h_{2n+1}(t)]$$

$$= \sum_{n=0}^{\infty} e^{at}h(2nT + t) + \sum_{n=0}^{\infty} e^{at}h((2n+2)T - t)$$

which becomes

$$\sum_{n=0}^{\infty} e^{at} h_n(t) = e^{at} h(t) + \sum_{n=1}^{\infty} e^{at} [h(2nT + t) + h(2nT - t)] \tag{2.20}$$

The right-hand side of (2.20) may be rewritten in terms of $f(t)$ as

$$\sum_{n=0}^{\infty} e^{at} h_n(t) = f(t) + \underbrace{\sum_{n=1}^{\infty} e^{-2anT} [f(2nT + t) + e^{2at} f(2nT - t)]}_{= \epsilon(t) \text{ (error term)}} \tag{2.21}$$

It can be shown (see below) that the error term can be made arbitrarily small for $t \in [0, T/2]$. As a consequence of this for such a range of t

$$f(t) \approx \frac{2}{T} e^{at} \left[\frac{1}{2} \operatorname{Re}[F(a)] + \sum_{k=1}^{\infty} \operatorname{Re}[F(a + i\omega_k)] \cos(\omega_k t) \right] \tag{2.22}$$

which, as observed in [3], amounts to a trapezoidal rule approximation to (2.7). Of course, the infinite series in (2.22) must in practice be truncated leading to an additional source of approximation error. The convergence properties of such a truncated series can be expected to yield the Gibbs phenomenon in the vicinity of discontinuities in $f(t)$, which is a fact also noted in [3]. An integral form of the truncated series approximation appears in [3], but this will not be repeated here. However, this integral form is similar to Equation (3.144) in [6] as it involves the Dirichlet kernel function.

For convenience define

$$\alpha_k = \begin{cases} \frac{1}{2} \operatorname{Re}[F(a)] & , \ k = 0 \\ \operatorname{Re}[F(a + i\omega_k)] & , \ k = 1, 2, 3, \ldots \end{cases} \tag{2.23}$$

in which case (2.22) may be rewritten as

$$f(t) \approx \frac{2}{T} e^{at} \sum_{k=0}^{\infty} \alpha_k \cos(\omega_k t) \tag{2.24}$$

or since $\alpha_k \in \mathbf{R}$ for all k

$$f(t) \approx \frac{2}{T} e^{at} \sum_{k=0}^{\infty} \operatorname{Re} \left[\alpha_k e^{i\omega_k t} \right] \tag{2.25}$$

We now wish to approximate $f(t)$ at the sample points $t = j\Delta t$ for $j = 0, 1, \ldots, M$. Hence

$$f(j\Delta t) \approx \frac{2}{T} e^{aj\Delta t} \operatorname{Re} \left[\sum_{k=0}^{\infty} \alpha_k e^{i \frac{k\pi}{T} j\Delta t} \right] \tag{2.26}$$

The N-term truncated approximation to $f(j\Delta t)$ is therefore given by

$$f(j\Delta t) \;\approx\; f_N(j\Delta t) \;=\; \frac{2}{T} e^{aj\Delta t} \,\mathrm{Re}\left[\sum_{k\,=\,0}^{N-1} \alpha_k e^{i\frac{k\pi}{T} j\Delta t}\right] \tag{2.27}$$

Since $j \;\leq\; M$, $t_{max} \;=\; M\Delta t$ is the largest time for which we compute an estimate of $f(t)$. Upon recalling the restriction that $0 \;\leq\; t \;\leq\; T/2$ we also see that $M\Delta t \;\leq\; \frac{1}{2}T$. We may modify (2.27) so that $f_N(j\Delta t)$ may be computed using a fast Fourier transform (FFT) algorithm. To this end select T so that

$$\frac{\pi\Delta t}{T} \;=\; \frac{2\pi}{N} \tag{2.28}$$

which implies that

$$T \;=\; \frac{1}{2} N\Delta t \tag{2.29}$$

and also that

$$M \;\leq\; \frac{1}{4} N \tag{2.30}$$

If (2.28) is enforced then (2.27) is now

$$f_N(j\Delta t) \;=\; \frac{4}{\Delta t} e^{aj\Delta t} \,\mathrm{Re}\left[\underbrace{\frac{1}{N}\sum_{k\,=\,0}^{N-1} \alpha_k e^{i\frac{2\pi}{N}kj}}_{=\,A_j}\right] \tag{2.31}$$

which is the corrected version of Equation (20) in [3]. The set $\{A_j \mid j \;=\; 0, 1, \;\ldots\;, N-1\}$ corresponds to a common definition of the inverse discrete Fourier transform (IDFT) [4] of the set $\{\alpha_k \mid k \;=\; 0, 1, \;\ldots\;, N-1\}$. It is the computation of A_j that may be efficiently performed using an FFT algorithm. If as is common practice $N \;=\; 2^p$ (for $p \;=\; 1, 2, 3, \;\ldots\;$) then a radix-2 FFT algorithm is required. Note that FFT algorithms exist for any natural number N, including prime numbers. However, such algorithms usually require much more effort to program than the radix-2 algorithms.

Now we may consider the error term $\epsilon(t)$ defined in (2.21). Since $t \;\in\; [0, T/2]$ it is reasonable to assume that for some constant $C \;>\; 0$ we have $|f(t)| \;\leq\; C$. That is, it is reasonable to believe

[4]Such a definition is common in texts on digital signal processing. As well, the technical details of how to implement FFT algorithms may also often be found in such texts. A good example of such a text would be A. V. Oppenheim, R. W. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, 1989.

that $f(t)$ is bounded on any finite interval of interest to us [5]. Therefore,

$$|\epsilon(t)| \leq C \sum_{n=1}^{\infty} e^{-2aTn}(1 + e^{2at}) = C \frac{e^{-2aT}}{1 - e^{-2aT}}(1 + e^{2at})$$

or

$$|\epsilon(t)| \leq C \frac{e^{2at} + 1}{e^{2aT} - 1} = C\, e^{a(t-T)} \frac{\cosh\,(at)}{\sinh\,(aT)} \tag{2.32}$$

However, we can also more simply say that

$$|\epsilon(t)| \leq C \sum_{n=1}^{\infty} e^{-2aTn}(1 + e^{aT}) = C \frac{e^{aT} + 1}{e^{2aT} - 1} \approx Ce^{-aT} \tag{2.33}$$

where the final approximation holds for sufficiently large aT. This may be used to make a reasonable choice for $a > 0$ when T is specified, and we have some idea about the size of the bound C.

3. The Algorithm of Lee-Weeks

As in Section 2 we begin this section with a presentation of some background material relevant to the development of the Lee-Weeks algorithm.

The method of Lee-Weeks [1,2] works with *Laguerre function expansions* of the approximation to $f(t)$. Thus, we now need to know something about Laguerre polynomials. The *orthonormal Laguerre functions* (Kreyszig [7]) are

$$e_n(t) = e^{-t/2}\, L_n(t) \tag{3.1}$$

for $n = 0, 1, 2, \ldots$ with $L_n(t)$ the *degree n Laguerre polynomial*, and these functions are orthonormal according to

$$\int_0^{\infty} e_n(t) e_m(t)\, dt = \delta_{n-m} \tag{3.2}$$

where *Kronecker's delta sequence* is defined to be

$$\delta_n = \begin{cases} 1 & , \; n = 0 \\ 0 & , \; n \neq 0 \end{cases} \tag{3.3}$$

Implicit in all of this is that the functions are defined for $t \in [0, \infty)$. From (3.1) into (3.2)

$$\int_0^{\infty} e^{-t} L_n(t) L_m(t)\, dt = \delta_{n-m} \tag{3.4}$$

[5] However, in [3] the assumption that $|f(t)| \leq Ct^m$ for some constant $C > 0$, and any $m = 0, 1, 2, \ldots$ is also made.

We see that the Laguerre polynomials are orthonormal with respect to the weighting function $w(t) = e^{-t}u(t)$. The first few Laguerre polynomials may be stated as follows:

$$L_0(t) = 1 \ , \ L_1(t) = 1 - t \ , \ L_2(t) = 1 - 2t + \frac{1}{2}t^2 \tag{3.5}$$

Various means exist for generating Laguerre polynomials of higher degrees and these are summarized in Kreyszig [7]. The most important method is a recurrence relation that will be considered a little later on. In general, it is worth noting that

$$L_n(0) = e_n(0) = 1 \tag{3.6}$$

for all n.

We will also need to know more properties of the Laplace transform. First of all, *Parseval's theorem* for the Laplace transform may be stated as

$$\int_0^\infty e^{-2at}|f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^\infty |F(a + i\omega)|^2 \, d\omega \tag{3.7}$$

Parseval theorems are associated with many other transforms, and in effect relate the energy/power content of a signal in the time-domain to that in the frequency (transform) domain. The result (3.7) may be justified as follows. From (2.5)

$$f(t) = \frac{e^{at}}{2\pi}\int_{-\infty}^\infty e^{i\omega t}F(a + i\omega) \, d\omega$$

so therefore (asterisk superscript "*" denotes complex conjugation)

$$\begin{aligned}
\int_0^\infty e^{-2at}f(t)f^*(t) \, dt &= \frac{1}{2\pi}\int_0^\infty e^{-at}f(t)\left[\int_{-\infty}^\infty e^{-i\omega t}F^*(a + i\omega) \, d\omega\right] dt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_0^\infty e^{-(a + i\omega)t}f(t) \, dt\right] F^*(a + i\omega) \, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty F(a + i\omega)F^*(a + i\omega) \, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty |F(a + i\omega)|^2 \, d\omega
\end{aligned}$$

which assumes that the order of integration may be changed as required. This derivation is illustrative of how to obtain Parseval theorems for many other transforms.

Now suppose that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ then for $\alpha > 0$

$$\mathcal{L}\{f(\alpha t)\} = \int_0^\infty e^{-st} f(\alpha t)\, dt = \frac{1}{\alpha}\int_0^\infty e^{-s\tau/\alpha} f(\tau)\, d\tau \quad (\tau = \alpha t)$$

and so

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \tag{3.8}$$

which is the scaling property for the Laplace transform. In addition to this property consider

$$\mathcal{L}\{e^{\alpha t} f(t)\} = \int_0^\infty e^{-(s-\alpha)t} f(t)\, dt = F(s - \alpha) \tag{3.9}$$

Of particular importance in what follows is $\mathcal{L}\{L_n(t)\}$. The necessary result may be derived as follows. From [7] we have

$$L_n(t) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} t^j \tag{3.10}$$

Observe that

$$\int_0^t L_{n-1}(\tau)\, d\tau = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \binom{n-1}{k-1} t^k \tag{3.11}$$

Next, observe that after some straighforward algebra

$$L_{n-1}(t) - \int_0^t L_{n-1}(\tau)\, d\tau =$$

$$1 + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] t^k + \frac{(-1)^n}{n!}\binom{n-1}{n-1} t^n \tag{3.12}$$

From Gould [8] (see p. iv)

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} \tag{3.13}$$

This immediately implies that (3.12) becomes

$$L_{n-1}(t) - \int_0^t L_{n-1}(\tau)\, d\tau = L_n(t) \tag{3.14}$$

Now since another property of the Laplace transform is

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s} F(s) \tag{3.15}$$

(integration property) we observe that upon Laplace transforming (3.14) we obtain

$$\mathcal{L}\{L_{n-1}(t)\} \; - \; \frac{1}{s}\,\mathcal{L}\{L_{n-1}(t)\} \; = \; \mathcal{L}\{L_n(t)\}$$

or

$$\mathcal{L}\{L_n(t)\} \; = \; \frac{s \; - \; 1}{s}\mathcal{L}\{L_{n-1}(t)\} \tag{3.16}$$

As $L_0(t) \; = \; u(t)$ (unit step function; recall that the functions at hand are assumed zero-valued for $t \; < \; 0$) we have $\mathcal{L}\{L_0(t)\} \; = \; 1/s$, and so (3.16) implies (via mathematical induction; see Zarowski [6] for a review of this method of proof)

$$\mathcal{L}\{L_n(t)\} \; = \; \frac{(s \; - \; 1)^n}{s^{n+1}} \tag{3.17}$$

This result often appears in mathmatical tables of the Laplace transform (e.g., CRC tables [9]).

The $N + 1$-term Laguerre series expansion that approximates $f(t)$ is given by

$$f_N(t) \; = \; e^{at} \sum_{n\,=\,0}^{N} a_n \, e_n \left(\frac{t}{T}\right) \tag{3.18}$$

where $T \; > \; 0$ is a suitable scaling factor. The Laguerre series expansion coefficients are given by

$$a_n \; = \; \frac{1}{T} \int_0^\infty e^{-at} f(t) \; e_n \left(\frac{t}{T}\right) \; dt \tag{3.19}$$

It is important to note that for all $\epsilon \; > \; 0$ there is an integer $N(\epsilon) \; > \; 0$ such that for any $N \; > \; N(\epsilon)$

$$\int_0^\infty e^{-2at} \; |f(t) \; - \; f_N(t)|^2 \; dt \; < \; \epsilon \tag{3.20}$$

This prescribes the sense in which $f_N(t)$ approximates $f(t)$. Plainly, the Laplace transform of the approximation is

$$F_N(s) \; = \; \int_0^\infty e^{-st} f_N(t) \; dt \tag{3.21}$$

However, if we employ (3.17) in combination with the basic Laplace transform properties (3.8), and (3.9) it is immediately apparent that we have a simple closed-form expression for this transform, which is

$$F_N(s) \; = \; \sum_{n\,=\,0}^{N} a_n \, \frac{\left(s - a - \frac{1}{2T}\right)^n}{\left(s - a + \frac{1}{2T}\right)^{n+1}} \tag{3.22}$$

Now from Parseval's theorem in (3.7)

$$\int_0^\infty e^{-2at}|f(t) - f_N(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(a+i\omega) - F_N(a+i\omega)|^2\, d\omega < \epsilon \tag{3.23}$$

$(N > N(\epsilon))$. From (3.22) with $s = a + i\omega$, and (3.23) we see that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| F(a+i\omega) - \sum_{n=0}^{N} a_n \frac{\left(i\omega - \frac{1}{2T}\right)^n}{\left(i\omega + \frac{1}{2T}\right)^{n+1}} \right|^2 d\omega < \epsilon \tag{3.24}$$

$(N > N(\epsilon))$.

Next, we make a change of the frequency variable ω according to

$$\omega = \frac{1}{2T}\cot\left(\frac{\theta}{2}\right) \tag{3.25}$$

We now digress for a moment to consider the *bilinear transformation* ($\alpha \in \mathbf{R}$)

$$z = \frac{s - \alpha}{s + \alpha} \tag{3.26}$$

which helps to explain (3.25). It is perhaps worth mentioning here that the bilinear transformation is the foundation for an important classical method of designing digital filters based on the transformation of analog prototype filters specified by a Laplace transfer function to corresponding target digital filters prescribed by a z-transfer function. In general, $s, z \in \mathbf{C}$. However, suppose here that $s = i\omega$ (so s is a point on the imaginary axis of the complex s-plane). In this situation there exists $\theta \in \mathbf{R}$ such that

$$e^{i\theta} = \frac{i\omega - \alpha}{i\omega + \alpha} \tag{3.27}$$

That is, the mapping (3.26) takes a point on the imaginary axis of the s-plane and maps it to a point on the unit circle $\{z \mid |z| = 1\}$ of the z-plane. In fact, from algebra involving suitable trigonometric identities we may use

$$\omega = \alpha\cot\left(\frac{\theta}{2}\right) \tag{3.28}$$

For our present circumstances we have $\alpha = 1/2T$.

Now note that we may rewrite (3.24) as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left| F(a+i\omega)\left(i\omega + \frac{1}{2T}\right) - \sum_{n=0}^{N} a_n \left(\frac{i\omega - \frac{1}{2T}}{i\omega + \frac{1}{2T}}\right)^n \right|^2}{\left| i\omega + \frac{1}{2T}\right|^2} d\omega < \epsilon \tag{3.29}$$

$(N > N(\epsilon))$. With (3.25), and noting that $\frac{d}{dx}\cot x = -\csc^2 x$ we now find that

$$\frac{1}{\left|i\omega + \frac{1}{2T}\right|^2} d\omega = \frac{1}{\left(\frac{1}{2T}\right)^2 \cot^2\left(\frac{\theta}{2}\right) + \left(\frac{1}{2T}\right)^2} \times \frac{-1}{4T\sin^2\left(\frac{\theta}{2}\right)} d\theta$$

Hence in (3.29)

$$\frac{1}{\left|i\omega + \frac{1}{2T}\right|^2} d\omega = T\, d\theta$$

Thus, (3.29) simplifies yielding

$$\frac{T}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{1}{2T} + \frac{i}{2T}\cot\left(\frac{\theta}{2}\right)\right) F\left(a + \frac{i}{2T}\cot\left(\frac{\theta}{2}\right)\right) - \sum_{n=0}^{N} a_n e^{i\theta n}\right|^2 d\theta < \epsilon \tag{3.30}$$

$(N > N(\epsilon))$. In other words

$$A(\theta) = \left[\frac{1}{2T} + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right] F\left(a + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right) \approx \sum_{n=0}^{N} a_n e^{i\theta n} \tag{3.31}$$

This expression will be used to find the expansion coefficients a_n which will then produce the desired approximation to $f(t)$ via the application of (3.18).

We may express $F(a + i\omega)$ in terms of its real and imaginary parts, i.e.,

$$F(a + i\omega) = F_r(a + i\omega) + i\, F_i(a + i\omega) \tag{3.32}$$

Therefore,

$$\begin{aligned} h(\theta) &= \mathrm{Re}\left\{\left[\frac{1}{2T} + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right] F\left(a + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right)\right\} \\ &= \frac{1}{2T}F_r\left(a + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right) - \frac{1}{2T}\cot\left(\frac{\theta}{2}\right) F_i\left(a + i\frac{1}{2T}\cot\left(\frac{\theta}{2}\right)\right) \\ &\approx \sum_{n=0}^{N} a_n \cos(n\theta) \end{aligned} \tag{3.33}$$

via (3.31). For

$$\theta_k = \frac{2k + 1}{N + 1}\frac{\pi}{2}, \quad k = 0, 1, \ldots, N \tag{3.34}$$

it can be shown that

$$a_n = \begin{cases} \frac{1}{N+1}\sum_{k=0}^{N} h(\theta_k) & , \; n = 0 \\ \frac{2}{N+1}\sum_{k=0}^{N} h(\theta_k)\cos(n\theta_k) & , \; n \neq 0 \end{cases} \tag{3.35}$$

This is actually *Chebyshev interpolation* to recover a_n from samples of $h(\theta)$. It is recommended by Weeks [2], and is based on the observation that $x = \cos\theta$, and so $T_n(x) = \cos(n \cos^{-1} x)$ is the *degree n Chebyshev polynomial of the first kind* (Zarowski [6], Kreyszig [7]).

An alternative interpolation procedure is DFT/FFT interpolation. Recall (3.31), and consider it for

$$\theta_k = \frac{2\pi}{N+1} k , \quad k = 0, 1, \ldots, N \tag{3.36}$$

yielding

$$A(\theta_k) = \sum_{n=0}^{N} a_n \exp\left(i \frac{2\pi}{N+1} kn\right) \tag{3.37}$$

This is essentially an $N+1$-point DFT. Hence a_n may be recovered from the samples $A(\theta_k)$ using an FFT algorithm. If $N = 2^p - 1$ then a radix-2 FFT algorithm may be employed.

Weeks [2] offers guidance in the selection of the constants a, and T. $L_n(x)$ has n real, distinct and positive-valued zeros x_k (i.e., $L_n(x_k) = 0$ for $k = 1, 2, \ldots, n$). If x_n is the largest such zero then it turns out that

$$x_n < 2n+1 + \sqrt{(2n+1)^2 + 1/4} \approx 4n$$

implying that $e_n(x)$ oscillates for $x \in [0, 4n]$, and decays exponentially for $x >> 4n$. This leads us to expect that the approximation to $f(t)$ given by $f_N(t)$ will only be a good one if for $t \in [0, t_{max}]$ we have

$$\frac{t_{max}}{T} < 4N \tag{3.38}$$

Weeks suggests that

$$T = \frac{t_{max}}{N} \tag{3.39}$$

Weeks also suggests that N be in the range $20 \leq N \leq 50$ and that

$$a = \left(a_0 + \frac{1}{t_{max}}\right) u \left(a_0 + \frac{1}{t_{max}}\right) \tag{3.40}$$

where as before $u(x)$ is the unit step function, and a_0 is the smallest real value such that for $a > a_0$ the following integrals converge:

$$\int_0^\infty e^{-at} |f(t)| \, dt < \infty , \quad \int_0^\infty e^{-2at} |f(t)|^2 \, dt < \infty \tag{3.41}$$

The constant a_0 is determined by finding the location of the right-most singularity of $F(s)$.

Additionally, Weeks [2] presents simple recursive schemes to determine such parameters as

$$\omega_j = \frac{1}{2T} \cot\left(\frac{\theta_j}{2}\right)$$

and $\cos(n\theta_j)$ for $j = 0, 1, 2, \ldots, N$. The development of these follows from certain trigonometric identities, and is summarized as follows.

From (3.34), $\theta_0 = \frac{1}{N+1}\frac{\pi}{2}$, and for $j = 1, 2, \ldots, N$

$$\theta_j = \theta_{j-1} + \frac{1}{N+1}\pi = \theta_{j-1} + 2\theta_0 \tag{3.42}$$

Since

$$\cot(\alpha + \beta) = \frac{\cot\alpha \cot\beta - 1}{\cot\alpha + \cot\beta} \tag{3.43}$$

we see that

$$\begin{aligned} \omega_j &= \frac{1}{2T}\cot\left(\frac{\theta_j}{2}\right) = \frac{1}{2T}\cot\left[\frac{\theta_{j-1}}{2} + \frac{1}{N+1}\frac{\pi}{2}\right] \\ &= \frac{1}{2T}\frac{\cot\left[\frac{\theta_{j-1}}{2}\right]\cot\left[\frac{1}{N+1}\frac{\pi}{2}\right] - 1}{\cot\left[\frac{\theta_{j-1}}{2}\right] + \cot\left[\frac{1}{N+1}\frac{\pi}{2}\right]} \\ &= \frac{\frac{1}{2T}\cot\left[\frac{\theta_{j-1}}{2}\right]\left(\frac{1}{2T}\cot\theta_0\right) - \frac{1}{4T^2}}{\frac{1}{2T}\cot\left[\frac{\theta_{j-1}}{2}\right] + \frac{1}{2T}\cot\theta_0} \end{aligned} \tag{3.44}$$

Define for convenience the constants

$$\gamma = \frac{1}{2T}\cot\theta_0 \,, \quad \delta = \frac{1}{4T^2} \tag{3.45}$$

and so thus we see that (3.44) reduces to

$$\omega_j = \frac{\gamma\omega_{j-1} - \delta}{\omega_{j-1} + \gamma} \tag{3.46}$$

for $j = 1, 2, \ldots, N$, where $\omega_0 = \frac{1}{2T}\cot\left(\frac{\theta_0}{2}\right) = \frac{1}{2T}\frac{1+\cos\theta_0}{\sin\theta_0}$. If we recall the identities

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3.47a}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{3.47b}$$

then from (3.42)

$$\cos\theta_j = \cos(2\theta_0)\cos\theta_{j-1} - \sin(2\theta_0)\sin\theta_{j-1} \tag{3.48a}$$

$$\sin\theta_j = \sin(2\theta_0)\cos\theta_{j-1} + \cos(2\theta_0)\sin\theta_{j-1} \tag{3.48b}$$

for $j = 1, 2, \ldots, N$ [6]. As well

$$\cos(n\theta_j) = \cos[(n-1)\theta_j + \theta_j] = \cos(n-1)\theta_j \cos\theta_j - \sin(n-1)\theta_j \sin\theta_j \tag{3.49}$$

and

$$\cos((n-1)\theta_j) = \cos(n\theta_j)\cos\theta_j + \sin(n\theta_j)\sin\theta_j$$

implying that

$$\cos((n-2)\theta_j) = \cos(n-1)\theta_j \cos\theta_j + \sin(n-1)\theta_j \sin\theta_j \tag{3.50}$$

Upon combining (3.49) with (3.50) we obtain

$$\cos(n\theta_j) + \cos((n-2)\theta_j) = 2\cos(n-1)\theta_j \cos\theta_j$$

or finally

$$\cos(n\theta_j) = 2\cos(n-1)\theta_j \cos\theta_j - \cos(n-2)\theta_j \tag{3.51}$$

for $n = 2, 3, \ldots, N$ (which is needed in (3.35)).

Finally, in order to determine $e_n(t/T)$ in (3.18) we require the following recursive method for $e_n(x)$ ([2], or [7]):

$$e_n(x) = \frac{2n - 1 - x}{n} e_{n-1}(x) - \frac{n-1}{n} e_{n-2}(x), \quad n > 1 \tag{3.52}$$

where $e_0(x) = e^{-x/2}$, $e_1(x) = (1-x)e^{-x/2}$.

4. Additional Comments

Hurwitz and Zweifel [10] have proposed that Fourier integrals be evaluated using Gaussian quadrature [6]. Now the method in Dubner and Abate [3] reveals that the problem of inverting Laplace transforms is equivalent to working with Fourier integrals, except for the presence of an exponential factor. Thus, the scheme in [10] may be applied to the problem of the numerical inversion of Laplace transforms [7] . The Dubner-Abate algorithm is essentially a trapezoidal rule which is interesting in that de Balbine and Franklin [11] maintain that with respect to accuracy "... the trapezoidal rule is at least as good as any other rule of quadrature for infinite range integrals." This implies that the Dubner-Abate algorithm is not likely to be inferior in accuracy relative to the approach of Hurwitz-Zweifel, and yet the Dubner-Abate algorithm is much simpler.

[6] Remember as well that

$$\cos(2\theta_0) = 2\cos^2\theta_0 - 1, \quad \sin(2\theta_0) = 2\sin\theta_0\cos\theta_0$$

[7] This has been suggested by Schmittroth; see reference [1] in Dubner and Abate [3].

However, a straightforward implementation of quadratures can suffer from slow convergence in that many terms are required to achieve good accuracy. Fourier integrals are typically oscillatory in their integrands, and as noted in Davis and Rabinowitz [12] an approach to the evaluation of such integrals is to compute the positive and negative contributions individually, and then sum the resulting infinite series. A problem with this is that one must know or be readily able to determine the locations of the zeros of the integrand, and of course this may be difficult in practice. As well the resulting series may itself be slow to converge. Transformation strategies (e.g., Euler's tranformation) can be used to accelerate convergence, and some more recent improvements to numerical methods for the inversion of Laplace transforms make use of ideas such as these. (Some of the papers cited in the present report refer to appropriate references in this area.)

The notion that quadratures of a higher order than the trapezoidal rule may not lead to significant improvements in accuracy has been challenged by Lee and Beaulieu [13]. To some extent [13] revisits the Poisson summation analysis of de Balbine and Franklin [11]. Therefore, it seems appropriate to give a brief account of the contents of [13] here.

The continuous-time (analog) Fourier transform of $f(t)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \tag{4.1}$$

The trapezoidal rule approximation is

$$F(\omega) \approx h \sum_{n=-\infty}^{\infty} f(nh)e^{-i\omega nh} = F(\omega) + E_t(\omega) \tag{4.2}$$

($h = \Delta t$) so $E_t(\omega)$ is the error term. From Papoulis [14] (Equation (3-87)) we note the *Poisson summation formula*

$$h \sum_{n=-\infty}^{\infty} f(nh)e^{-inh\omega} = \sum_{n=-\infty}^{\infty} F(\omega - \omega_s n) \tag{4.3}$$

where $\omega_s = 2\pi/h$ (sampling frequency in radians/second if time t is in units of seconds). Thus,

$$E_t(\omega) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} F(\omega - \omega_s n) \tag{4.4}$$

It is apparent that $E_t(\omega)$ is *aliasing error.* If h is small enough, and $F(\omega)$ is *bandlimited* so that $f(t)$ has no energy for frequencies $\omega > \omega_B$, then $E_t(\omega) = 0$ provided that $\omega_s > 2\omega_B$ (Whittaker-Shannon-Nyquist sampling theorem). Of course, many practical situations (such as the computational problems motivating this report) do not yield a bandlimited $f(t)$.

As in [13] let $T_h[\omega]$ denote the trapezoidal rule approximation to $F(\omega)$, and let $S_h[\omega]$ denote the Simpson's rule approximation to $F(\omega)$. Of course, the subscript "h" denotes the panel width in the approximations. It is known via the Romberg procedure (e.g., [6]) that the two approximations relate according to

$$S_h[\omega] \;=\; \frac{1}{3}(4T_h[\omega] \;-\; T_{2h}[\omega]) \tag{4.5}$$

Since from (4.2) $T_h[\omega] \;=\; F(\omega) \;+\; E_t(\omega)$, via (4.5) we have

$$\begin{aligned} S_h[\omega] \;&=\; F(\omega) \;+\; \frac{1}{3}\left\{ 4 \sum_{\substack{n\,=\,-\infty \\ n\,\neq\,0}}^{\infty} F(\omega \;-\; n\omega_s) \;-\; \sum_{\substack{n\,=\,-\infty \\ n\,\neq\,0}}^{\infty} F(\omega \;-\; \frac{1}{2}n\omega_s) \right\} \\ &=\; F(\omega) \;+\; E_s(\omega) \end{aligned}$$

where

$$E_s(\omega) \;=\; \frac{1}{3}\sum_{n\,=\,1}^{\infty}\left\{4[F(\omega - n\omega_s) \;+\; F(\omega + n\omega_s)] \;-\; \left[F\left(\omega - \frac{1}{2}n\omega_s\right) \;+\; F\left(\omega + \frac{1}{2}n\omega_s\right)\right]\right\} \tag{4.6}$$

It is argued in [13] (and examples are provided) showing that signals $f(t)$ exist for which the error term $E_s(\omega)$ is smaller than the error term $E_t(\omega)$. That is, the Simpson's rule quadrature can be more accurate than the trapezoidal rule quadrature, even though the Simpson's rule quadrature is of a higher order than the trapezoidal rule.

5. Examples

In this final section we illustrate the algorithms of Dubner-Abate and Lee-Weeks as applied to the following special cases, all of which were examples considered in [2] and/or [3]:

$$f_1(t) \;=\; \frac{2}{\sqrt{3}}\exp\left(-\frac{t}{2}\right)\,\sin\left(\frac{\sqrt{3}}{2}t\right)\,u(t) \;\longleftrightarrow\; F_1(s) \;=\; \frac{1}{s^2 \;+\; s \;+\; 1}$$

$$f_2(t) \;=\; u(t \;-\; 25) \;\longleftrightarrow\; F_2(s) \;=\; \frac{\mathrm{e}^{-25s}}{s}$$

$$f_3(t) \;=\; \frac{1}{2}\,t\sin(t)\,u(t) \;\longleftrightarrow\; F_3(s) \;=\; \frac{s}{(s^2 \;+\; 1)^2}$$

where $u(t)$ is the unit step function. Respectively, we shall refer to these three examples as the "damped sinusoid," "shifted step," and the "unbounded sinusoid." Note that in both [2] and [3] the function $f_1(t)$ is not correctly stated.

The application of the Dubner-Abate algorithm to the above three examples appears in Figs. 1 to 3. In all cases the numerically approximated inverse transform is a very good fit to the true inverse transform. The Lee-Weeks approximations appear in Figs. 4 to 7. Note that these results were generated using T as given in (3.39), with $t_{max} = M\Delta t$, and that $a = 1/t_{max}$ (which assumes that $a_0 = 0$ in (3.40)). The approximation errors are certainly larger than the approximation errors associated with the Dubner-Abate method. The Gibbs phenomenon is quite evident in the example of Fig. 5. Figures 6 and 7 compare Chebyshev interpolation to DFT/FFT interpolation in the Lee-Weeks algorithm. The results suggest that Chebyshev interpolation is to be preferred over DFT/FFT interpolation, and that the Lee-Weeks methodology is sensitive to the choice of interpolation procedure. Additional experiments (not reported here) seem to confirm this conclusion. However, Weeks [2] does not consider anything other than Chebyshev interpolation. Overall the results in Figs. 1 to 6 are certainly comparable to those reported in [2], and [3].

The examples considered here seem to imply that the Dubner-Abate algorithm is more accurate than the Lee-Weeks algorithm. However, a consideration of all examples to be found in [2], [3] and [4] suggests that this is a premature conclusion. Springer [5] has remarked that there seems to be no one universally superior method for the numerical inversion of Laplace transforms. More recent literature on the subject of numerical inversion of Laplace transforms suggests that although improvements in algorithms have been made since the 1960s that there is still no one universally superior approach.

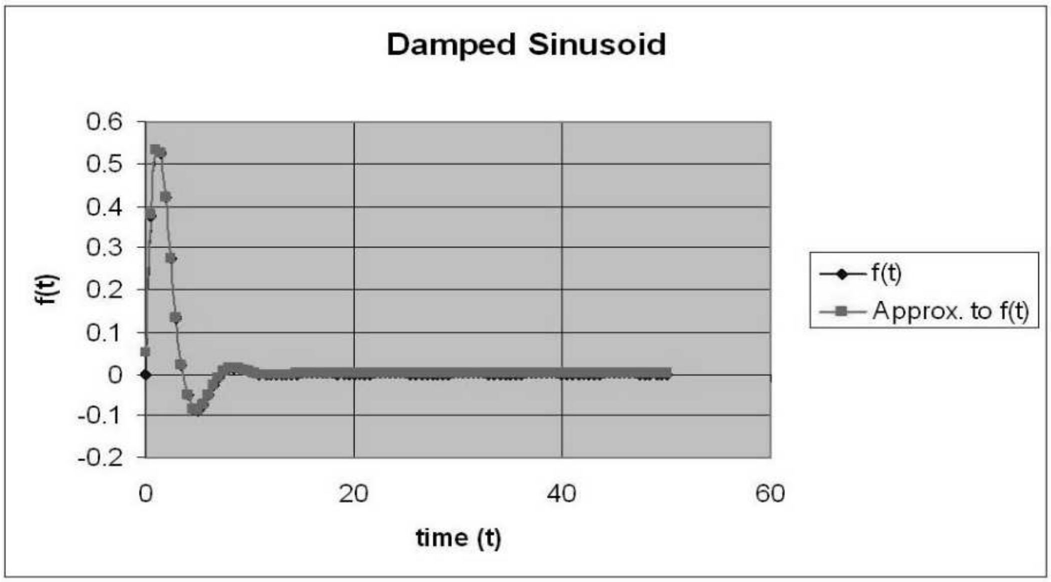

Figure 1: Inverse transform of the damped sinusoid using Dubner-Abate. Here $N = 512$, $M = 100$, $\Delta t = 0.5$, $a = 0.05$.

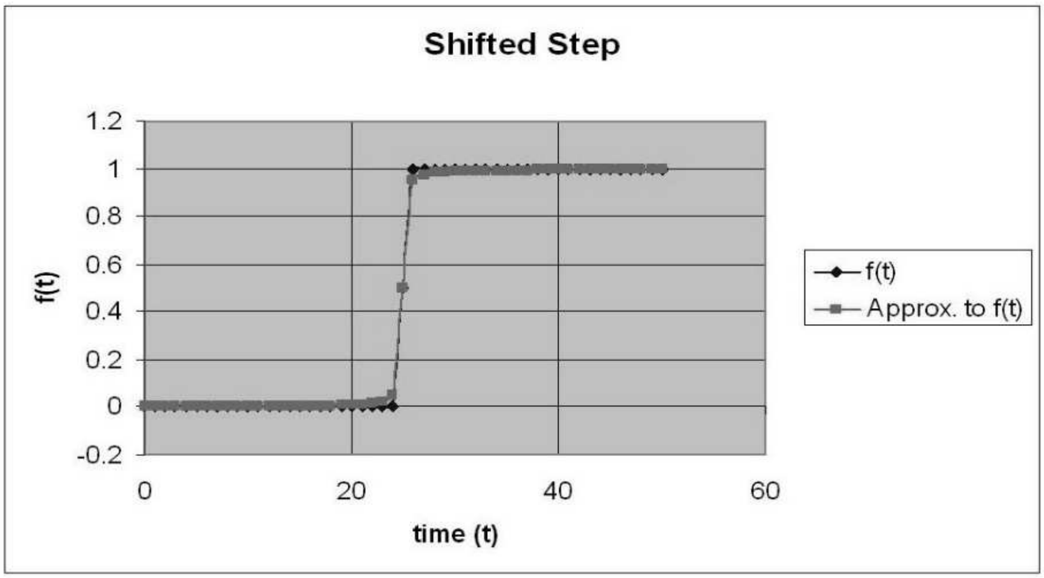

Figure 2: Inverse transform of the shifted step using Dubner-Abate. Here $N = 256$, $M = 50$, $\Delta t = 1.0$, $a = 0.05$.

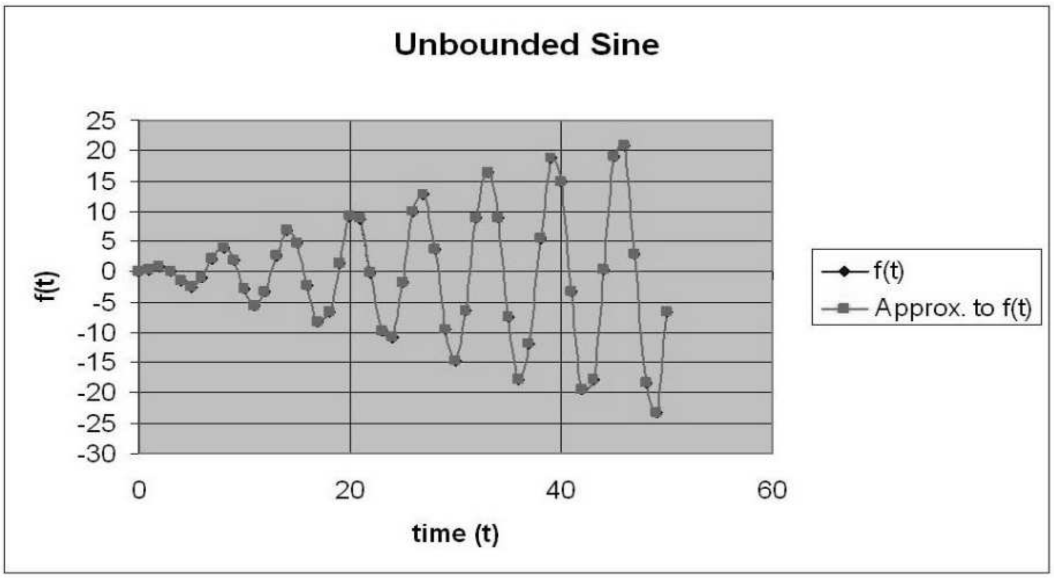

Figure 3: Inverse transform of the unbounded sine using Dubner-Abate. Here $N = 256$, $M = 50$, $\Delta t = 1.0$, $a = 0.05$.

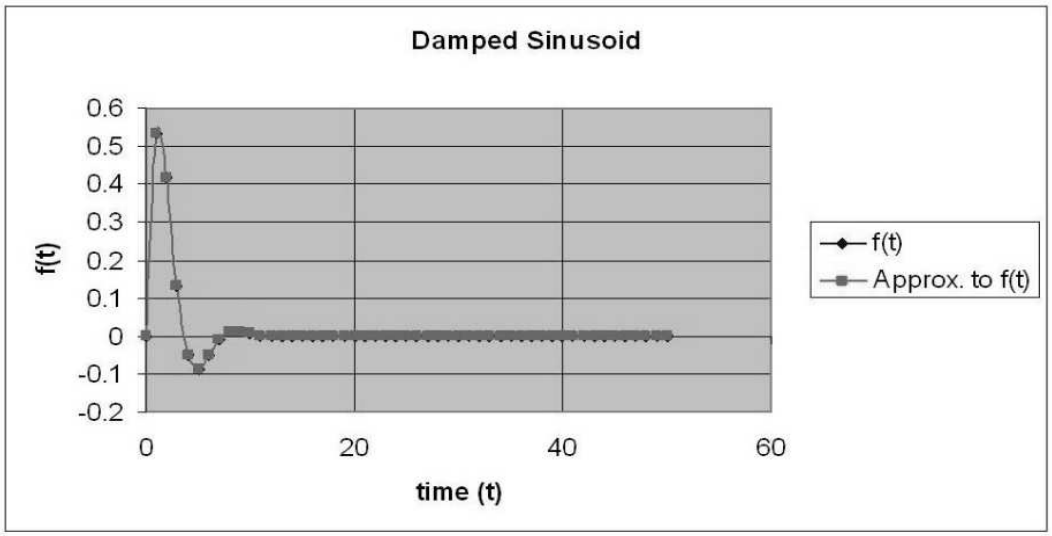

Figure 4: Inverse transform of the damped sinusoid using Lee-Weeks with Chebyshev interpolation. Here $N = 31$, $M = 50$, $\Delta t = 1.0$.

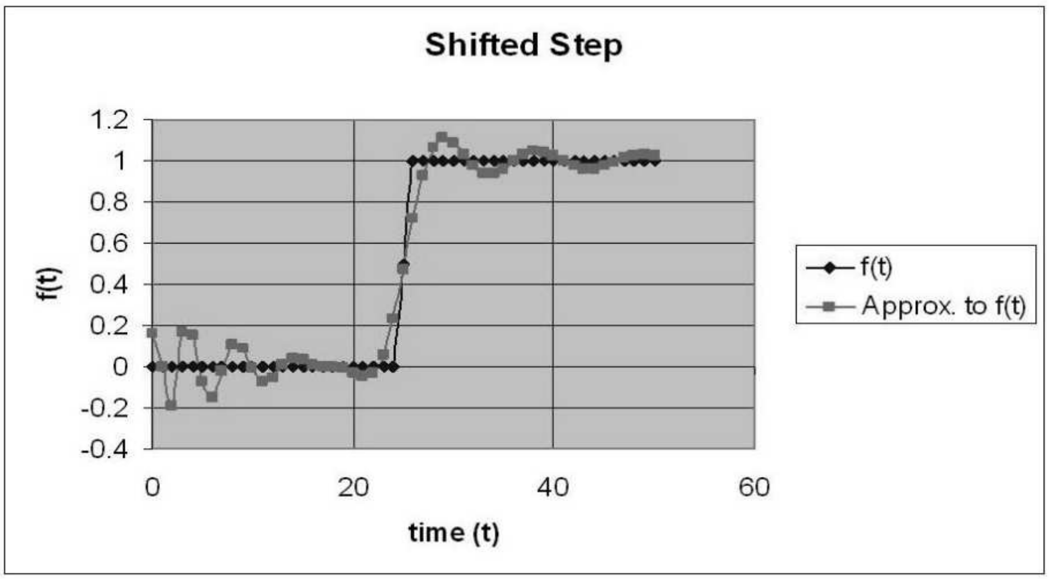

Figure 5: Inverse transform of the shifted step using Lee-Weeks with Chebyshev interpolation. Here $N = 31$, $M = 50$, $\Delta t = 1.0$.

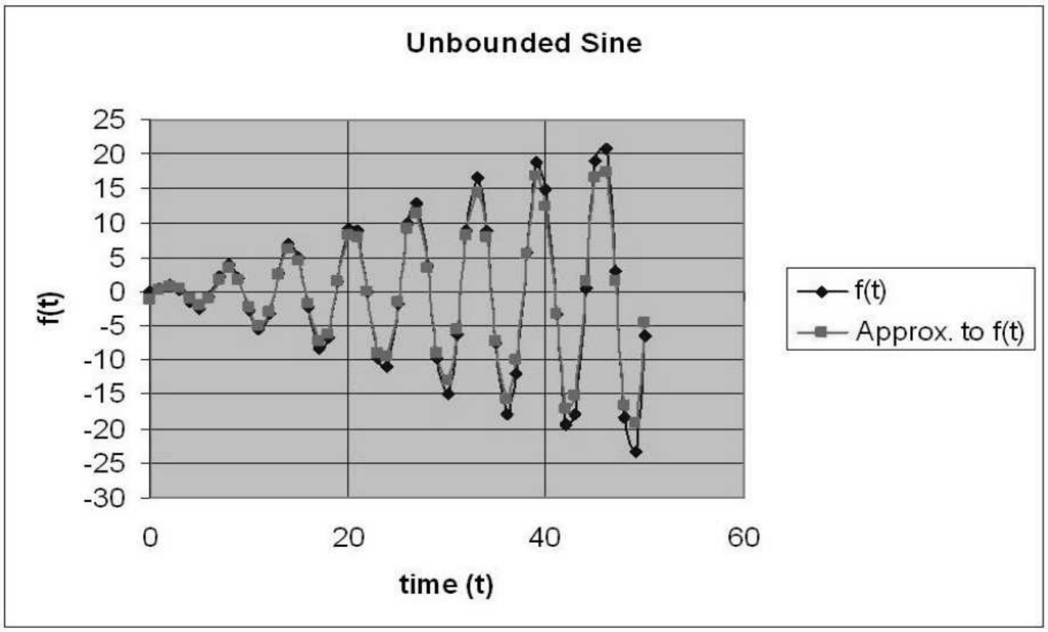

Figure 6: Inverse transform of the unbounded sine using Lee-Weeks with Chebyshev interpolation. Here $N = 127$, $M = 50$, $\Delta t = 1.0$.

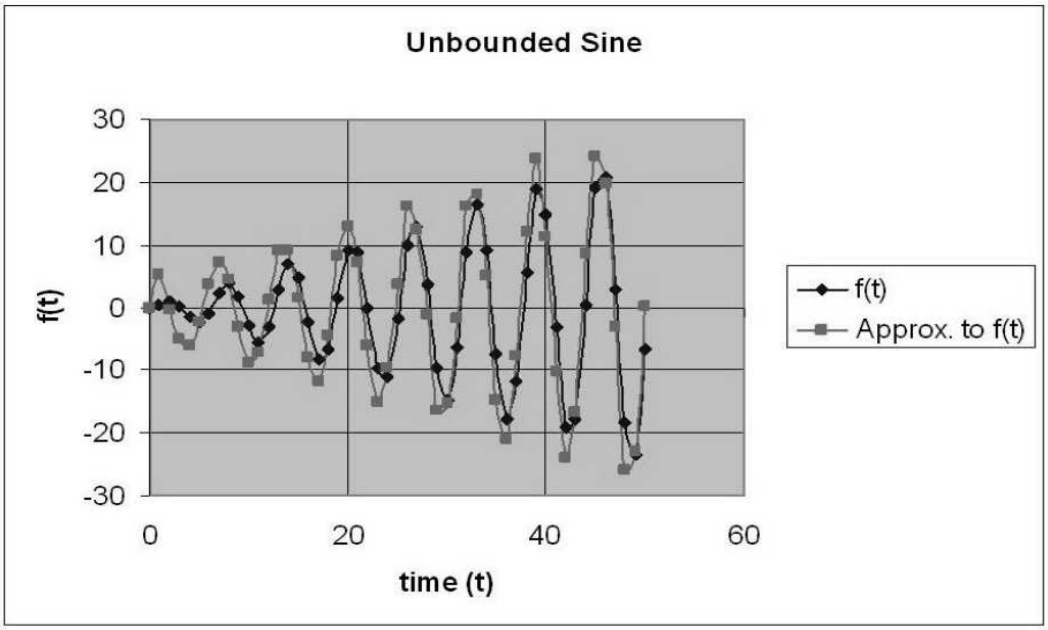

Figure 7: Inverse transform of the unbounded sine using Lee-Weeks with DFT/FFT interpolation. Here $N = 127$, $M = 50$, $\Delta t = 1.0$.

Acknowledgment

This report was prepared with the aid of many different software tools. Numbers were "crunched" using Java J2SE with the BlueJ IDE. That is, the algorithms of Dubner-Abate and Lee-Weeks were implemented as Java methods. The Java FFT method of Palmer [15] was used. Plots were initially generated using the MS Excel charting feature. These were converted by various means (details omitted) into encapsulated postscript (eps) form so that they may be incorporated into a LaTeX file. The LaTeX source file was processed under Linux 9 into postscript (ps) form, and finally into pdf form.

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